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# Synchronization using dynamic coupling

Lutz Junge and Ulrich Parlitz*
*Drittes Physikalisches Institut, Universität Göttingen, Bürgerstraße 42-44, D-37073 Göttingen, Germany*
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A systematic coupling procedure is introduced for synchronizing arbitrary chaotic dynamical systems. This coupling exploits the existing contraction properties of the flow and surpresses divergence only along those directions in state space, where the underlying flow is not contracting. In this way, systems can be synchronized using a minimum of transmitted information for guaranteed high-quality synchronization. Applications in combination with sporadic driving and in partitioned state spaces are numerically illustrated.

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Synchronization of periodic signals is a well-known phenomenon in science and engineering, and even chaotic systems may be coupled in a way such that their oscillations are synchronized [1]. If the coupled systems are very similar or (almost) identical, their state vectors $\mathbf{x}$ and $\mathbf{y}$ converge during a synchronization transient to the same (chaotic) trajectory. The fact that chaotic systems may synchronize despite their sensitive dependence on initial conditions can be explained by a surpression of expanding dynamics in state space transversal to the synchronization manifold $\mathbf{x}=\mathbf{y}$ due to the coupling. Conventional coupling schemes are mainly based on global coupling forces and make no systematic use of the contraction properties of the underlying flow. We introduce in the following, a way to design a coupling for arbitrary pairs of (identical) systems that suppresses exponential divergence of the dynamics of the synchronization error $\mathbf{x}-\mathbf{y}$, and fully exploits the contraction properties of the flow of the given systems.

*Dynamic coupling.* We shall introduce the dynamic coupling scheme, first for discrete dynamical systems, and then for continuous systems. Let

$$\mathbf{x}_{n+1}=f(\mathbf{x}_n) \quad \text{and} \quad \mathbf{y}_{n+1}=f(\mathbf{y}_n)$$

be two identical $m$-dimensional chaotic dynamical systems that we want to synchronize by means of a suitable coupling mechanism. Since the systems are assumed to be chaotic, any pair of orbits $\{\mathbf{x}_n\}$ and $\{\mathbf{y}_n\}$ starting at closely neighboring initial conditions $\{\mathbf{x}_0\}$ and $\{\mathbf{y}_0\}$ will (exponentially) diverge. As long as the distance between the orbits is small the evolution of the synchronization error $\mathbf{e}_n=\mathbf{y}_n-\mathbf{x}_n$ is governed by the linear system

$$\mathbf{e}_{n+1}=\mathrm{D}f(\mathbf{y}_n)\cdot\mathbf{e}_n,$$

where $\mathrm{D}f(\mathbf{y}_n)$ denotes the Jacobian matrix of $f$ at $\mathbf{y}_n$. The increase of the error $\mathbf{e}_n$, i.e., the divergence of orbits can best be analyzed and described in terms of the singular value decomposition (SVD) of the Jacobian matrix

$$\mathrm{D}f=U\cdot W\cdot V^{tr},$$

where $U$ and $V$ are orthogonal matrices and $W=\mathrm{diag}(w_i)$ is a diagonal matrix with positive elements $w_i$, the singular values (SVs) of $\mathrm{D}f$. The matrix $\mathrm{D}f(\mathbf{y}_n)$ depends on the current state as the matrices of the SVD do, but here, and in the following, we shall indicate this dependence explicitly only in cases where misunderstandings have to be avoided. Let us assume now that at $\mathbf{y}_n$ locally $k$ noncontracting directions exist, that are given by the column vectors $\mathbf{v}_1,\ldots,\mathbf{v}_k$ of the matrix $V$ with corresponding SVs $w_1,\ldots,w_k\geqslant 1$. Our goal is to design a coupling matrix $C=C(\mathbf{y}_n)$ that surpresses the local expansion of the flow along these directions. We shall first consider the case of unidirectional coupling where the desired coupling can be written as

$$\mathbf{y}_{n+1}=f(\mathbf{y}_n)+C(\mathbf{x}_n-\mathbf{y}_n), \tag{1}$$

yielding an error dynamics

$$\mathbf{e}_{n+1}=[\mathrm{D}f-C]\mathbf{e}_n.$$

Choosing

$$C=U\cdot\mathrm{diag}(w_1,\ldots,w_k,0,\ldots,0)\cdot V^{tr}, \tag{2}$$

the matrix $\mathrm{D}f-C$ governing the error dynamics is given by $U\cdot\mathrm{diag}(0,\ldots,0,w_{k+1},\ldots,w_m)\cdot V^{tr}$ and thus possesses only SVs that are smaller than one. The choice of this matrix $C$ guarantees the linear stability of the synchronized state. Using this coupling matrix $C$, the coupling term can be rewritten

$$C[\mathbf{x}_n-\mathbf{y}_n]=\sum_{i=1}^{k} w_i[\langle\mathbf{x}_n,\mathbf{v}_i\rangle-\langle\mathbf{y}_n,\mathbf{v}_i\rangle]\mathbf{u}_i, \tag{3}$$

where $\langle,\rangle$ denotes the standard scalar product. If $\mathbf{y}_n$ is close to $\mathbf{x}_n$, the matrix $V(\mathbf{y}_n)$ of the SVD of $\mathrm{D}f(\mathbf{y}_n)$ can be approximated [2] by the matrix $V(\mathbf{x}_n)$ of the SVD of $\mathrm{D}f(\mathbf{x}_n)$, and we obtain the *dynamic coupling*

$$\mathbf{x}_{n+1}=f(\mathbf{x}_n),$$

$$\mathbf{y}_{n+1}=f(\mathbf{y}_n)+\sum_{i=1}^{k} w_i[s_i(\mathbf{x}_n)-s_i(\mathbf{y}_n)]\mathbf{u}_i, \tag{4}$$

*Email address: parlitz@dpi.physik.uni-goettingen.de

 

based on $i=1,\ldots,k$ scalar signals $s_i(\mathbf{x}_n)=\langle \mathbf{x}_n, \mathbf{v}_i(\mathbf{x}_n)\rangle$ and $s_i(\mathbf{y}_n)=\langle \mathbf{y}_n, \mathbf{v}_i(\mathbf{y}_n)\rangle$. The same reasoning yields for bidirectional coupling the scheme

$$\mathbf{x}_{n+1}=f(\mathbf{x}_n)+\frac{1}{2}\sum_{i=1}^{k} w_i(\mathbf{x}_n)[s_i(\mathbf{y}_n)-s_i(\mathbf{x}_n)]\mathbf{u}_i(\mathbf{x}_n),$$

$$\mathbf{y}_{n+1}=f(\mathbf{y}_n)+\frac{1}{2}\sum_{i=1}^{k} w_i(\mathbf{y}_n)[s_i(\mathbf{x}_n)-s_i(\mathbf{y}_n)]\mathbf{u}_i(\mathbf{y}_n).$$

For continuous dynamical systems, a similar result can be derived starting again from two uncoupled systems:

$$\dot{\mathbf{x}}=f(\mathbf{x}) \quad \text{and} \quad \dot{\mathbf{y}}=f(\mathbf{y}).$$

The evolution of the distance $\mathbf{e}=\mathbf{y}-\mathbf{x}$ between two neighboring orbits is given by the linearized equation

$$\dot{\mathbf{e}}=Df(\mathbf{y})\cdot\mathbf{e}. \tag{5}$$

For a small time step $\Delta t$, the solution of Eq. (5) may be approximated by $\mathbf{e}(\Delta t)=A\cdot\mathbf{e}(0)$, where $A(\Delta t)=I+\Delta t Df$ is an approximation of the linearized flow of the system, $I$ denotes the identity matrix, and $\mathbf{e}(0)$ is the initial condition. SVD of $A(\Delta t)=U\cdot W\cdot V^{tr}$ provides SVs $w_i(\Delta t)$ of $A(\Delta t)$ that may be interpreted in the same way as the SVs of the linearization of discrete systems, but still depend on the (arbitrary) time step $\Delta t$. To eliminate this dependence on $\Delta t$, we consider

$$A^{tr}\cdot A=I+\Delta t(Df^{tr}+Df)+\Delta t^2 Df^{tr}\cdot Df.$$

Neglecting terms of higher order in $\Delta t$ and defining

$$B=Df^{tr}+Df,$$

we obtain $I+\Delta tB=A^{tr}\cdot A=V\cdot W^2\cdot V^{tr}$ and thus $B=V\cdot D\cdot V^{tr}$ with $D=\mathrm{diag}(d_1,\ldots,d_m)=(W^2-I)/\Delta t$. The diagonal elements $d_i$ are the eigenvalues of the matrix $B$. Solving for the SVs of $A(\Delta t)$, we obtain $w_i(\Delta t)=\sqrt{1+\Delta t d_i}$. Contraction with $w_i<1$ thus occurs for those directions where $d_i<0$. This stability criterion will now be forced by introducing a suitable (unidirectional) coupling

$$\dot{\mathbf{y}}=f(\mathbf{y})+C(\mathbf{x}-\mathbf{y}).$$

The coupling matrix $C$ has to be chosen in a way that all eigenvalues of

$$\tilde{B}=Df^{tr}-C^{tr}+Df-C=B-(C^{tr}+C)$$

are negative. This goal can be achieved using a symmetric matrix $C$ that manipulates all non-negative eigenvalues $d_1,\ldots,d_k\geq 0$ of $B$:

$$C^{tr}+C=2C=2V\cdot\mathrm{diag}(c_1,\ldots,c_k,0,\ldots,0)\cdot V^{tr}$$

with $c_i>d_i/2$ for $i=1,\ldots,k$. This yields a matrix $\tilde{B}$ with negative SVs and results in the following coupling term:

$$C(\mathbf{x}-\mathbf{y})=\sum_{i=1}^{k} c_i[\langle \mathbf{x},\mathbf{v}_i\rangle-\langle \mathbf{y},\mathbf{v}_i\rangle]\mathbf{v}_i,$$

where $\mathbf{v}_i$ denotes again the column vectors of the orthogonal matrix $V$. Hence the coupling term is expressed in terms of some constants $c_i>d_i/2$ that are related to the non-negative eigenvalues $d_i$ of $B=Df^{tr}+Df$ and the corresponding eigenvectors $\mathbf{v}_i$, and depends in this way on the Jacobian of the vector field $Df$ along the orbit.

An analysis for bidirectionally coupled continuous systems yields the same form of the coupling matrix except that only half of the coupling strength for unidirectional is necessary to stabilize the synchronized state.

*Hénon map.* To demonstrate the efficiency of the proposed coupling (4), we shall now consider the $m=2$ dimensional Hénon map:

$$x^1_{n+1}=1-a(x^1_n)^2+bx^2_n,$$

$$x^2_{n+1}=x^1_n,$$

with $a=1.4$ and $b=0.3$. The SVs of the Jacobian

$$Df(\mathbf{x}_n)=\begin{pmatrix} -2ax^1_n & b \\ 1 & 0 \end{pmatrix}$$

are given by the square roots of the eigenvalues of the matrix $(Df)^{tr}\cdot Df$ and thus depend on $x^1$ only. For the first, SV holds $w_1\geq 1$, indicating expansion except for $x^1=0$. Since $w_2\leq b<1$ for all $x^1$, only a single noncontracting direction exists and the dynamic coupling is given by the scalar signal $s_1(\mathbf{x}_n)=\langle \mathbf{x}_n, \mathbf{v}_1(\mathbf{x}_n)\rangle$.

In order to avoid numerical artifacts, white noise (uniformly distributed random numbers) of amplitude $10^{-12}$ is added to the dynamical variables during all following simulations. Furthermore, the Hénon map possesses not only the chaotic Hénon attractor we want to study, but also orbits that diverge. Unfortunately, the chaotic Hénon attractor is located quite close to the basin of this coexisting attractor at infinity. Therefore, the coupling is not applied in those cases where it would kick the driven system towards a diverging solution.

The first question we want to address with the Hénon example is how much the approximation of the local singular values and vectors at the response system degrades the performance of this coupling. For this purpose we have compared the coupling given in Eq. (4) with a corresponding coupling (3) *without* approximations. Figure 1 shows the decrease of the synchronization error $e=\|\mathbf{x}-\mathbf{y}\|$ as a function of time $n$ for the proposed dynamic coupling using only a scalar time series $s_1(\mathbf{x}_n)$ (solid curve) and the ideal dynamic coupling (dotted curve), where the vector $\mathbf{v}_1(\mathbf{x}_n)$ also has to be transmitted to the response system. As can be seen only for synchronization errors below $10^{-4}$, both curves are significantly different and the convergence of the proposed coupling (4) becomes slower than that of the ideal coupling (3).

The efficiency of dynamic coupling allows for deactivating the coupling from time to time in order to reduce the information flow from the drive to the response system. We shall now discuss two ways to exploit this feature.

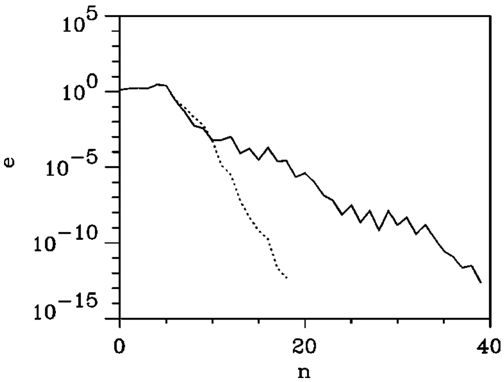


FIG. 1. Synchronization error $e=\|\mathbf{x}-\mathbf{y}\|$ vs time $n$ of the dynamic coupling (solid curve) and the ideal dynamic coupling (dotted curve).

*Sporadic driving.* An obvious way to reduce the information flow from drive to response is *sporadic driving* [3], where $T$ iterations are performed before the next coupling signal is computed from the current state and transmitted to the response system where it is applied in the coupling. Using dynamic coupling (4) for the Hénon system, synchronization in terms of negative conditional Lyapunov exponents occurs for $T\leqslant 7$, and high quality synchronization, without intermittent bursts of the synchronization error [4,5], can be achieved with $T\leqslant 5$. Thus only every fifth iterate of the Hénon map has to transmitted as a coupling signal to the response system, reducing the information flow considerably. Note that the commonly used coupling signal $s=x_1^2$ for the Hénon map has to be applied for each iteration to guarantee high-quality synchronization.

*Partition based coupling.* Another way to reduce the information flow of the dynamic coupling is motivated by the observation that the singular values $w_i$ of $\mathrm{D}f(\mathbf{x}_n)$ depend, in general, on the state $\mathbf{x}_n$. Therefore, one may restrict the coupling to those regions in the state space where strong expansion has to be surpressed. Since for the Hénon map the largest SV is a function of the first state component $x_1$, only (with a local minimum at $x_1=0$) one may activate coupling only if $|x_1|>x_c>0$, or equivalently, if $w_1>w_{min}$. Figures 2(a) and 2(b) show the resulting largest conditional Lyapunov exponent versus the coupling threshold $w_{min}$ and the percentage of iterations where coupling is activated, respectively. Figure 2(c) shows a histogram of the number of iterations between activation of the coupling for $w_{min}=2.8$.

The dynamic coupling as defined in Eq. (4) requires the repeated computation of the SVD of the Jacobian at different states. To avoid these computations, one may partition the state space such that in each cell the required singular values and vectors are approximated by their average values in the particular cell, which are stored in a database. The number of necessary cells depends, of course, on the dynamical system. In the case of the Hénon map, we achieve synchronization using a dynamical coupling based on a coarse partition with $2\times 2$ cells. Furthermore, one may restrict coupling again to those cells of the partition that are visited with a high probability and/or possess large SVs. Figure 3 shows a partition with $20\times 20$ cells. High-quality synchronization is already

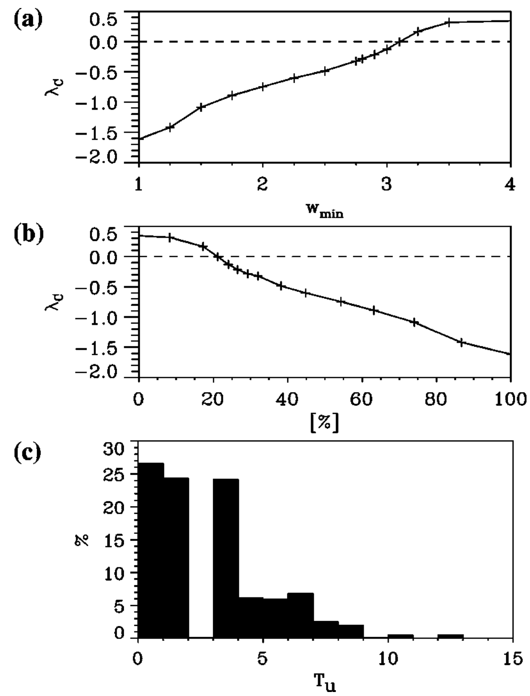


FIG. 2. Largest conditional Lyapunov exponent $\lambda_c$ vs SV threshold $w_{min}$ (a) and vs percentage of iterations where coupling is active (b). (c) Histogram of the number $T_u$ of iterations between two subsequent coupling events.

achievd if the ten most important cells (plotted as black and dark gray squares) are used in the coupling. For selecting these ten cells, the invariant measures $\mu(C_{ij})$ of all cells have been estimated and multiplied with the largest local singular value $w_1$. The ten cells possess the largest SV-weighted measures and cover about 25% of the total invariant measure. If instead of cell number ten, the 11th cell (according to our ordering) is used for implementing the coupling, strong intermittent bursts of desynchonization occur. Therefore, the proper location of cells is crucial when using a small number of coupling cells, only. Using a com-

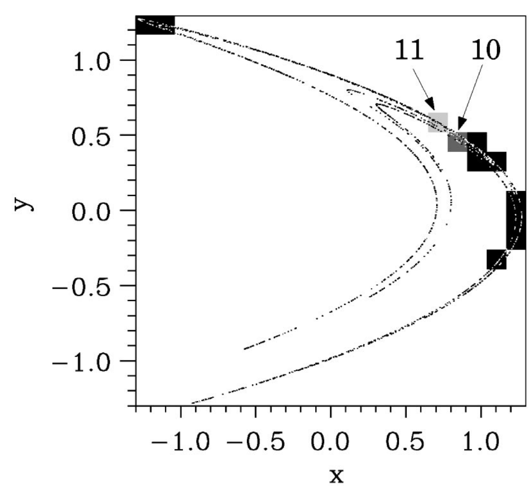


FIG. 3. Black squares indicate the location of cells where the dynamic coupling is activated.

mon start cell (to avoid long transients) and the configuration of coupling cells as a (discrete) key, one may in this way devise communication protocols where between the coupling signals encoded messages are transmitted that can be decoded using the state of the synchronized response system.

Coupling partitions may also be used with other coupling schemes. Furthermore, partitioning and constant approximations of singular values and vectors can also be combined with sporadic driving. In this case, coupling is activated exactly after $T$ iterations regardless of the current cell. Simulations with the Hénon map showed that for a partition with $20\times 20$ cells, sporadic driving with $T\leqslant 5$ leads to high quality synchronization.

*Delay embedding.* A drawback of the coupling scheme (4) is the fact that for each noncontracting direction, a coupling signal $s_i(\mathbf{x}_n)$ $(i=1,\dots,k)$ is required. To avoid the transmission of such $k$-dimensional coupling vectors, we shall extend the coupling now by some delay embedding procedure [6]. Let $h(\mathbf{x}_n)$ be a scalar observable such that the $d$-dimensional embedding $\mathbf{p}_n=(h(\mathbf{x}_n),h\circ f^{-1}(\mathbf{x}_n),\dots,h\circ f^{1-k}(\mathbf{x}_n))$ fulfills Takens' theorem [7]. Then a diffeomorphic delay embedding map $F$ exists that maps $\mathbf{x}_n$ to $\mathbf{p}_n=F(\mathbf{x}_n)$ and $\mathbf{y}_n$ to $\mathbf{q}_n=F(\mathbf{y}_n)=(h(\mathbf{y}_n),h\circ f^{-1}(\mathbf{y}_n),\dots,h\circ f^{1-k}(\mathbf{y}_n))$. Taylor expansion of the inverse $F^{-1}$ yields for neighboring states $\mathbf{x}_n$ and $\mathbf{y}_n$

$$\mathbf{x}_n-\mathbf{y}_n\approx DF^{-1}(\mathbf{q}_n)\cdot(\mathbf{p}_n-\mathbf{q}_n), \qquad (6)$$

where $DF^{-1}(\mathbf{q}_n)$ denotes the (pseudo) inverse of the Jacobian matrix $DF$ of $F$. The matrix $DF$ can be computed explicitly for given functions $h$ and $f$, and for its inversion again, SVD may be used yielding $DF=U_F\cdot W_F\cdot V_F^{tr}$ and $DF^{-1}=V_F\cdot W_F^{-1}\cdot U_F^{tr}$. Substituting Eq. (6) for the difference of states in Eq. (1), we obtain a coupling that depends on the present and past values of the scalar coupling signal $s_n=h(\mathbf{x}_n)$ entering the delay vector $\mathbf{p}_n$:

$$\mathbf{y}_{n+1}=f(\mathbf{y}_n)+C\cdot DF^{-1}(\mathbf{q}_n)\cdot(\mathbf{p}_n-\mathbf{q}_n).$$

Here the coupling matrix $C$ is designed as given in Eq. (2) to force convergence, and the matrix $DF^{-1}$ is used to translate differences of states from the reconstruction space to the original state space. In a similar way, bidirectional coupling may be defined, and for continuous systems, one could also use derivative coordinates.

*Conclusion.* A systematic coupling scheme has been presented that exploits ideally the contraction properties of the underlying flow and ensures linear stability of the synchronized state in all points of the state space. This approach is similar in spirit, but conceptionally different from other methods for chaos control and observer design [1,8]. Besides coupling using expansion directions given by singular vectors, we have also investigated schemes based on unstable eigenvectors. A comparison showed that the SVD-based coupling presented in this Rapid Communication is superior to methods using eigenvalues and eigenvectors.

Information about the local expanding directions can give important hints for the design of a global coupling function because it allows one to locate regions in the state space where a given coupling signal or scheme may fail due to strong expansion rates. On the other hand, an analysis of local expansion can also be used to evaluate the suitability of a given coupling signal for synchronization of two systems. Only if this signal contains significant components along the expanding direction during the time evolution, will it lead to synchronization when used in some (arbitrary) coupling mechanism.

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[1] Chaos **7** (1997); IEEE Trans. Circuits and Systems **44** (1997); *Handbook of Chaos Control*, edited by H. G. Schuster (Wiley-VCH, Weinheim, 1999).

[2] Such an approximation is not necessary for maps with constant Jacobian matrices as, for example, the cat map. In that case, already after the first iteration states are projected *exactly* onto the stable subspace, the synchronization error decreases exponentially to zero (in the absence of external perturbations).

[3] U. Parlitz *et al.*, Physica D **109**, 139 (1997); T. Stojanovski *et al.*, Phys. Rev. E **55**, 4035 (1997).

[4] R. Brown and N. Rulkov, Chaos **7**, 395 (1997).

[5] D. Gauthier and J. Bienfang, Phys. Rev. Lett. **77**, 1751 (1996).

[6] H. Huijberts, T. Lilge, and H. Nijmeijer, Phys. Rev. E **59**, 4691 (1999).

[7] F. Takens, *Detecting Strange Attractors in Turbulence*, edited by D.A. Rand and L.S. Young, Lecture Notes in Mathematics Vol. 898 (Springer-Verlag, New York, 1980), pp. 366–381; T. Sauer, J.A. Yorke, and M. Casdagli, J. Stat. Phys. **65**, 579 (1991).

[8] P. So, E. Ott, and W.P. Dayawansa, Phys. Rev. E **49**, 2650 (1994); Y.-C. Lai and C. Grebogi, *ibid.* **47**, 2357 (1993).